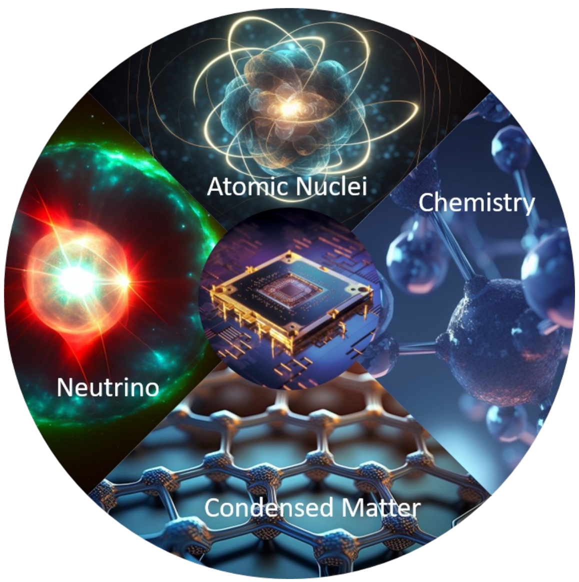
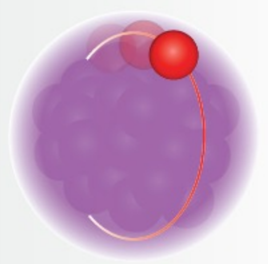
Exploring the richness of the Lipkin Model and its extensions: from nuclear to neutrino physics and quantum computing

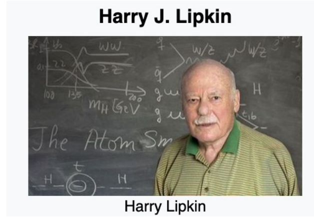
Denis Lacroix (IJCLab, Orsay, France)



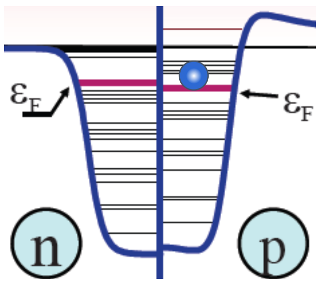
T. Ayral, P. Besserve, D. Lacroix, and E.A. Ruiz Guzman ,
Quantum computing with and for many-body physics, EPJA 59 (2023)

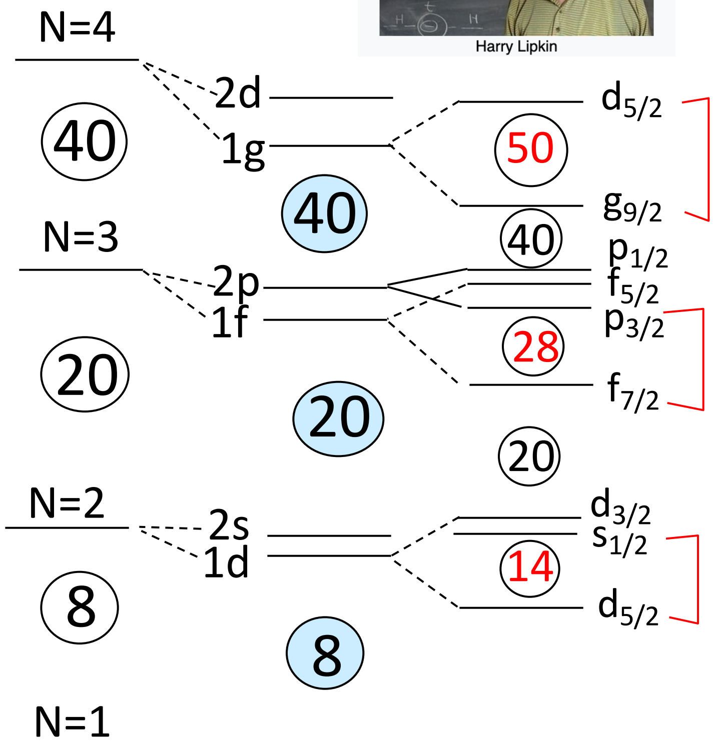
Lipkin model: its physical motivation in atomic nuclei



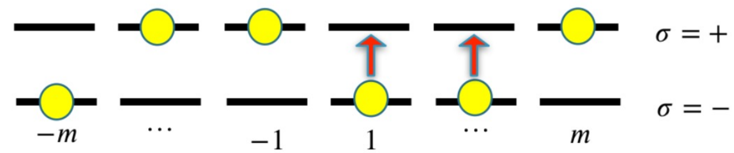


Assuming two shells with the same *j* quantum number each with *(2j+1)* states





H.O. + L^2 + $\vec{L}.\vec{S}$



$$H = \epsilon J_0 - \frac{1}{2}V(J_+^2 + J_-^2)$$

$$J_0 = \frac{1}{2}\sum_{\sigma,m} \sigma c^\dagger_{\sigma,m} c_{\sigma,m}$$

Counts difference of particle number

$$J_+ = J_-^\dagger = \sum_m c^\dagger_{1,m} c_{-1,m}$$

Make jumps between Lower and upper level

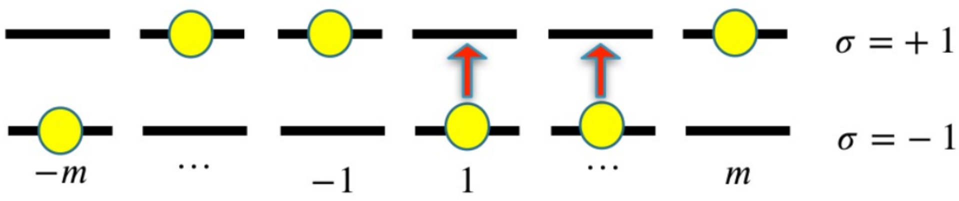
VALIDITY OF MANY-BODY APPROXIMATION METHODS FOR A SOLVABLE MODEL

(I). Exact Solutions and Perturbation Theory

H. J. LIPKIN,
Weizmann Institute of Science, Rehovoth, Israel
N. MESHKOV and A. J. GLICK †
Weizmann Institute of Science, Rehovoth, Israel
and
University of Maryland, College Park, Maryland ††

Received 18 February 1964

Abstract: In order to test the validity of various techniques and formalisms developed for treating many-particle systems, a model is constructed which is simple enough to be solved exactly in some cases, but yet is non-trivial. The construction of such models is based on the observation



$$H = \epsilon J_0 - \frac{1}{2}V(J_+^2 + J_-^2)$$

$$J_0 = \frac{1}{2}\sum_{\sigma,m} \sigma c^\dagger_{\sigma,m} c_{\sigma,m}$$

Counts difference of particle number

$$J_+ = J_-^\dagger = \sum_m c^\dagger_{1,m} c_{-1,m}$$

Make jumps between Lower and upper level

Conservation laws and symmetries

For a set of *N* 2-level systems:

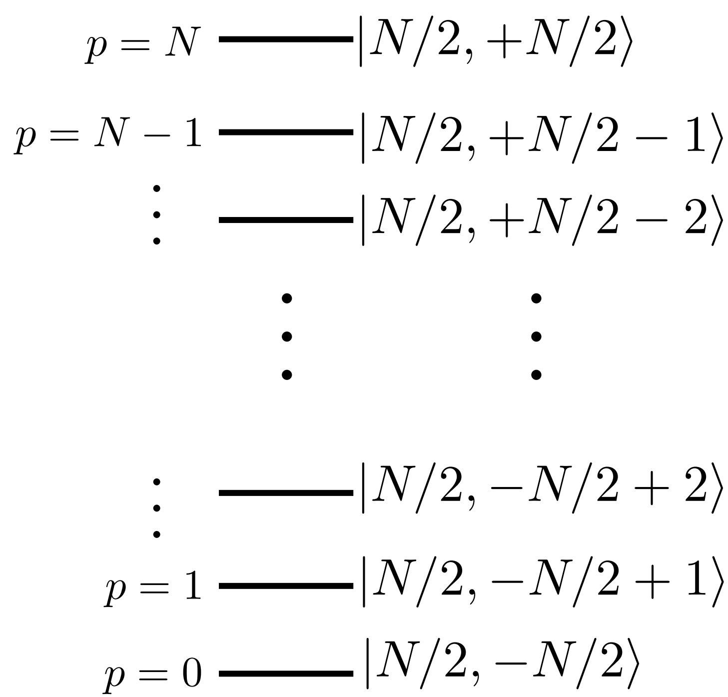
- Full Fock space has a size 2^{2N}
- Particle number conservation for *N* particles 2^N
- Permutation invariance

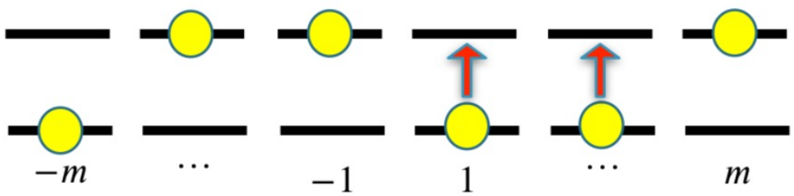
$$|p\rangle = \frac{1}{\sqrt{C_N^p}} \sum_{\mathcal{S}_p} |\text{Slater}\rangle$$

p denotes the number of particles in the upper or lower level

Permutation-invariant states are eigenstates of the total angular momentum $\mathbf{J^2}$ ➡ $(N+1)$ states $|J, M\rangle$

- Parity (odd/even M) ➡ $(N+1)/2$





$$H = \epsilon J_0 - \frac{1}{2} V (J_+^2 + J_-^2)$$

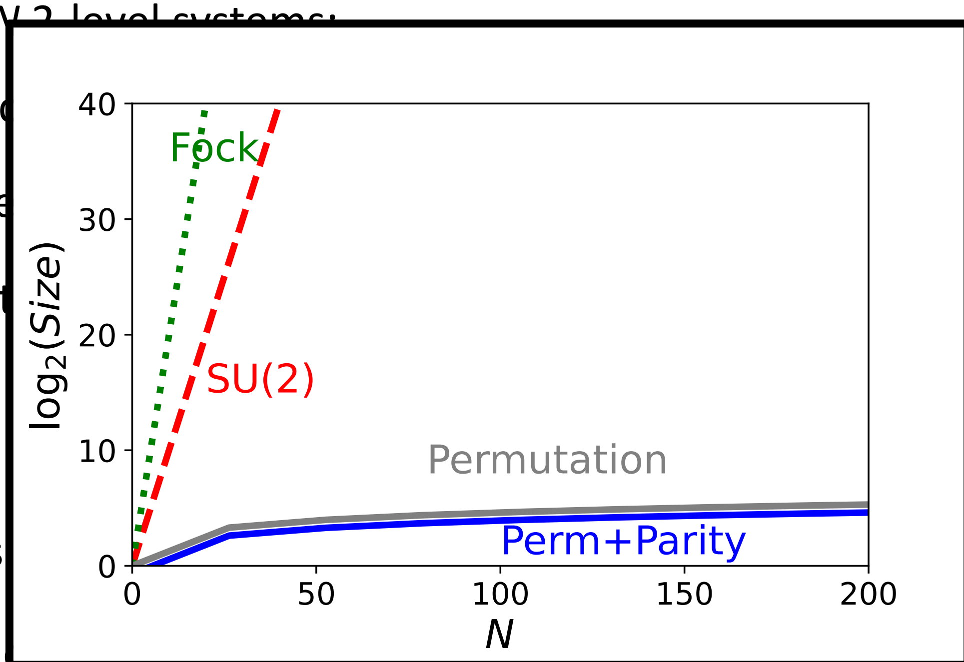
$$J_0 = \frac{1}{2} \sum_{\sigma,m} \sigma c^\dagger_{\sigma,m} c_{\sigma,m}$$ Counts difference of particle number

$$J_+ = J_-^\dagger = \sum_m c^\dagger_{1,m} c_{-1,m}$$ Make jumps between Lower and upper level

Conservation laws and symmetries

For a set of *N* 2-level systems:

- Full Fo
- Particle
- Permut



p denotes ... el

Permutati ... g
total angular momentum $\mathbf{J^2}$ ➡ $(N+1)$ states $|J, M\rangle$

- Parity (odd/even M) ➡ $(N+1)/2$

$p = N$ —— $|N/2, +N/2\rangle$

$p = N-1$ —— $|N/2, +N/2-1\rangle$

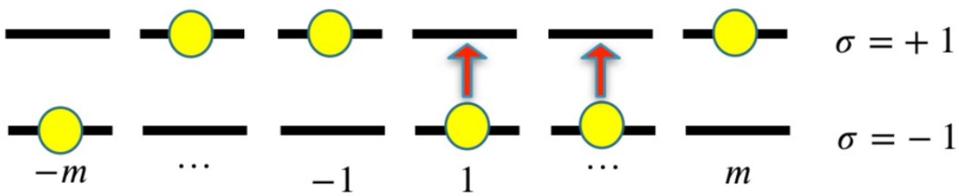
$\vdots$ —— $|N/2, +N/2-2\rangle$

$\vdots$ $\vdots$

$\vdots$ —— $|N/2, -N/2+2\rangle$

$p = 1$ —— $|N/2, -N/2+1\rangle$

$p = 0$ —— $|N/2, -N/2\rangle$



$$H = \epsilon J_0 - \frac{1}{2} V (J_+^2 + J_-^2)$$

Standard Angular momentum algebra

$p = N$ —— $|N/2, +N/2\rangle$

$p = N-1$ —— $|N/2, +N/2-1\rangle$

$\vdots$ —— $|N/2, +N/2-2\rangle$

$\vdots \qquad \vdots$

$\vdots$ —— $|N/2, -N/2+2\rangle$

$p = 1$ —— $|N/2, -N/2+1\rangle$

$p = 0$ —— $|N/2, -N/2\rangle$

$$\begin{aligned} J_0|J,M\rangle &= M|J,M\rangle \\ J_+|J,M\rangle &= \sqrt{J(J+1)-M(M+1)}|J,M+1\rangle \\ J_-|J,M\rangle &= \sqrt{J(J+1)-M(M-1)}|j,M-1\rangle \end{aligned}$$

$$\langle J, M'|H|J,M\rangle$$

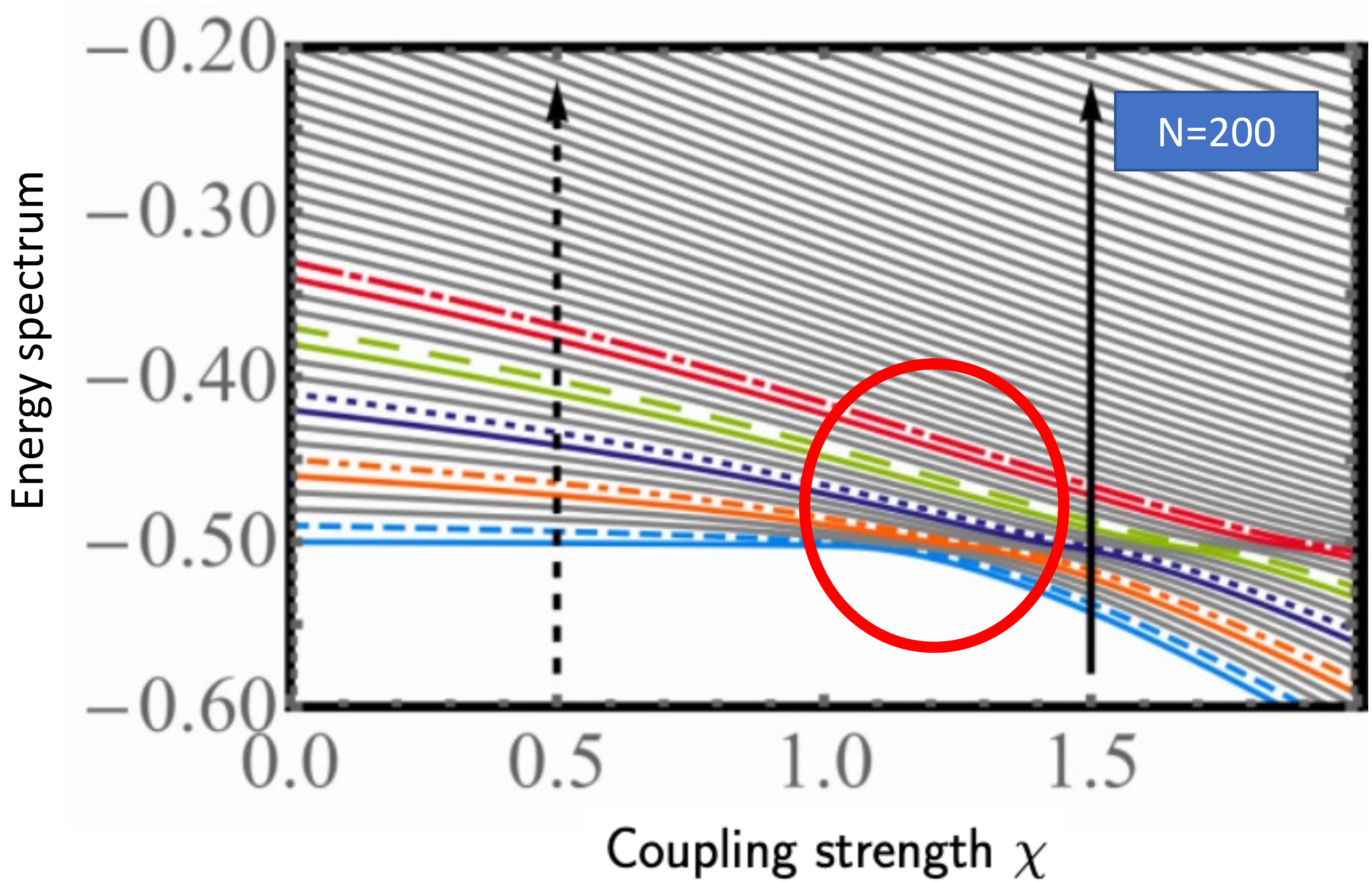
Brute-force diagonalization

$$\{\Psi_n, E_n\}$$

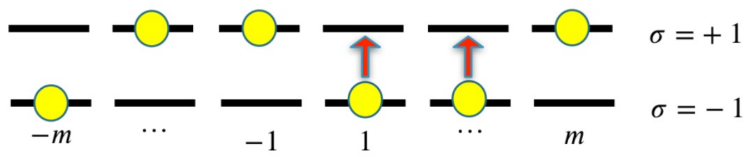
$$|\Psi_n\rangle = \sum_M c_M(n)|J,M\rangle$$

Hamiltonian

$$\frac{H}{\epsilon} = J_0 - \frac{1}{2(N-1)}\chi(J_+^2 + J_-^2)$$

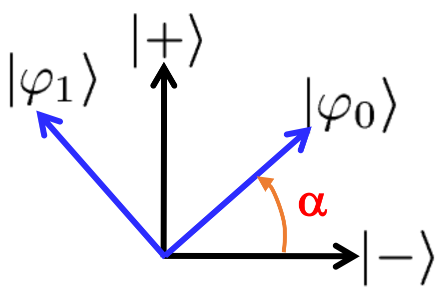


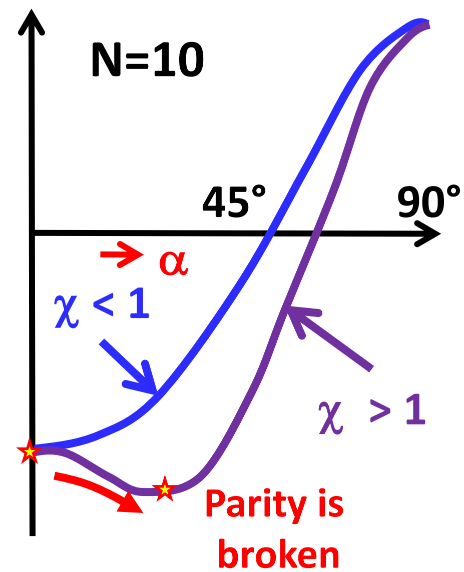
- It simulates the behaviour between degenerated energy levels between the Fermi surface
- Parity symmetry
- Number of particles symmetry

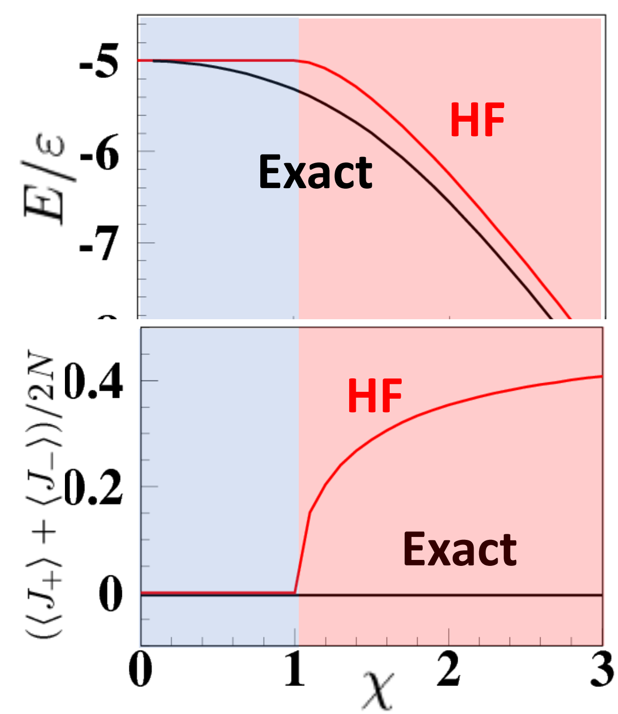


See for instance : Ring and Schuck book

Hartree-Fock solution







Energy

$$|\Phi\rangle = \Pi_{p=1}^{N} a^\dagger_{0,p}|-\rangle$$

$$\mathcal{E}_{MF}[\alpha,\varphi] = -\frac{\varepsilon N}{2}\left\{\cos(2\alpha) + \frac{\chi}{2}\sin^2(2\alpha)\cos(2\varphi)\right\}$$

with $\chi = \dfrac{V(N-1)}{\varepsilon}$

SU(2) coherent states

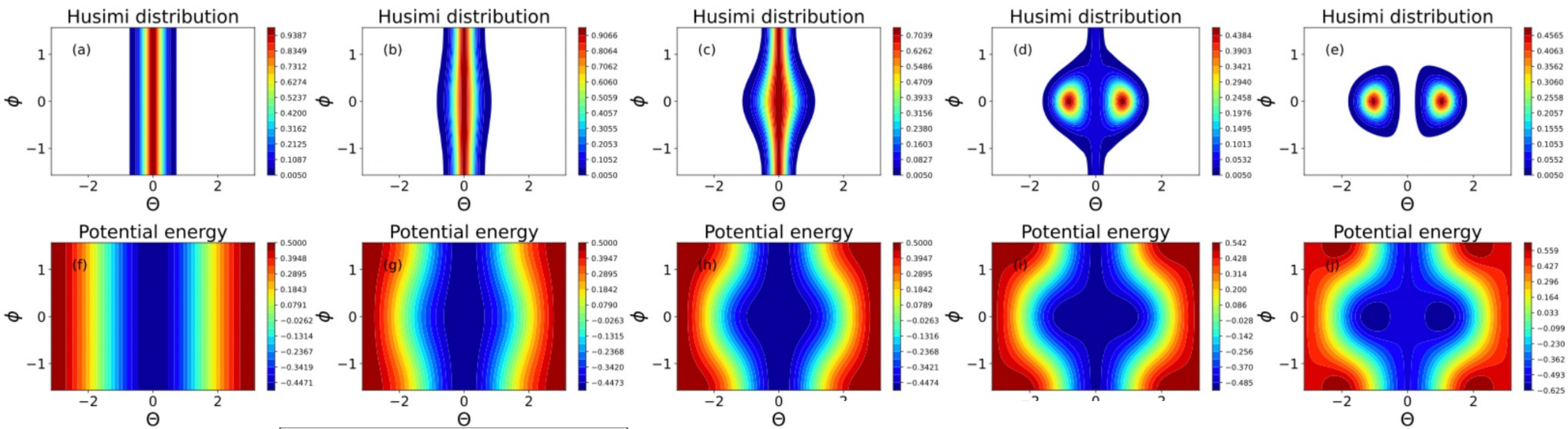
Hartree-Fock states are coherent SU(2) algebra

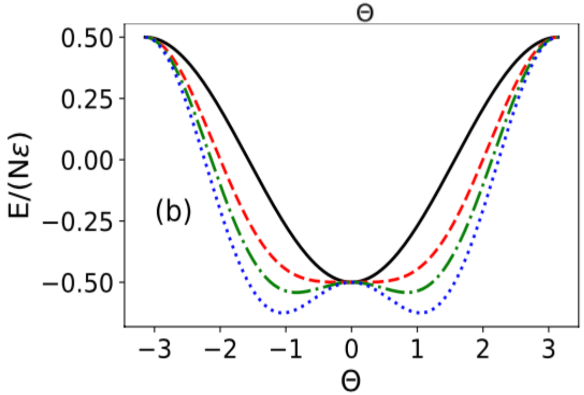
overcompleteness

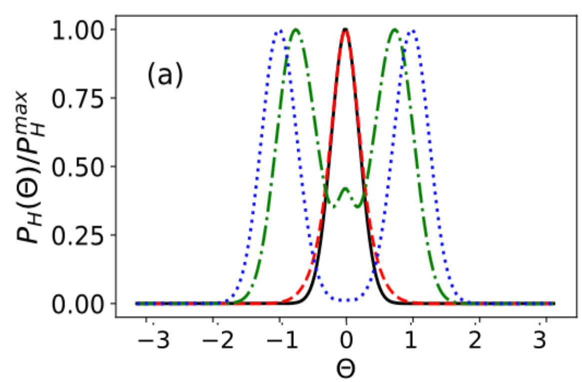
$$|\Omega,\theta\rangle = |\theta,\varphi\rangle \;=\; \frac{1}{(1+|z|^2)^{N/2}} e^{zJ_+}|J,-J\rangle \qquad z = \tan(\theta/2)e^{+i\phi}$$

$$\frac{N+1}{4\pi}\int |\Omega\rangle\langle\Omega| \mathrm{d}\Omega = 1$$

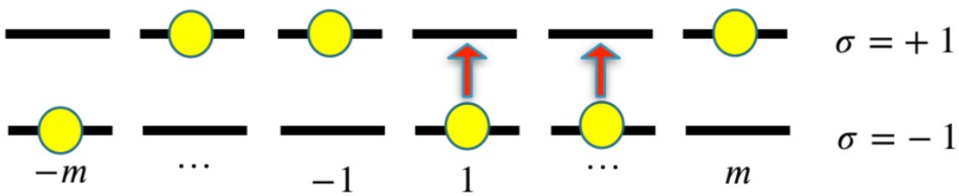
→ Husimi distribution $Q(\Omega,t) \;=\; \langle\Omega|D(t)|\Omega\rangle$





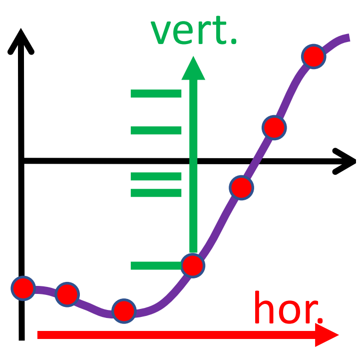


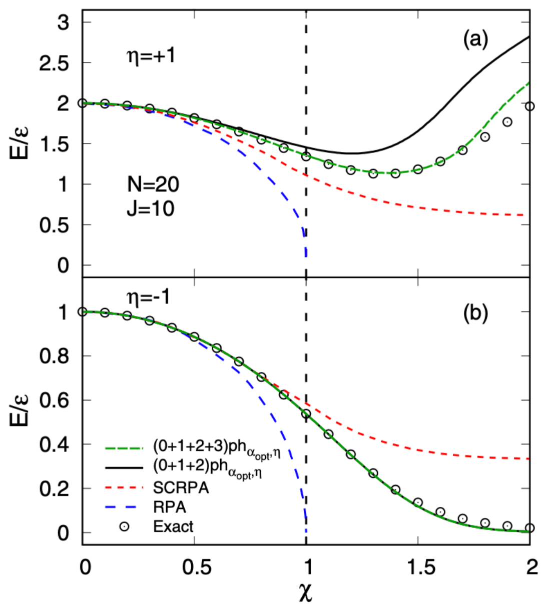
What can be done with it: firstly test many-body approximate theories



- Original Lipkin-Meshkov-Glick papers (pert. th.)
- Holzwarz NPA (76) [ATDHF, RPA, Multi-conf]

Multi-configuration (vert. vs horizontal)





Ripoche et al, PRC97(2018)

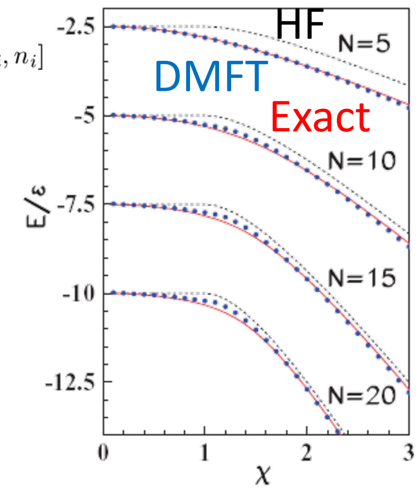
Density matrix Functional Theory

$$\mathcal{E}^{N}_{\rm Corr}[\varphi_i, n_i] = \eta(N)\frac{N(N-1)}{2}\,\mathcal{E}^{N=2}_{\rm Corr}[\varphi_i, n_i]$$

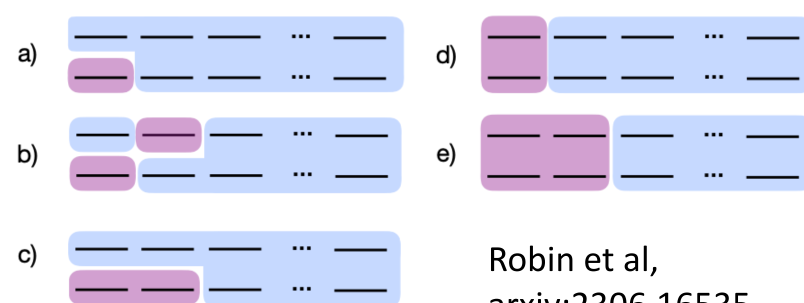
Large N limit
Dusuel, Vidal, PRL93 (2004)

DMFT for the Lipkin model

Lacroix, PRC79(2009)



Quantum Information and entanglement

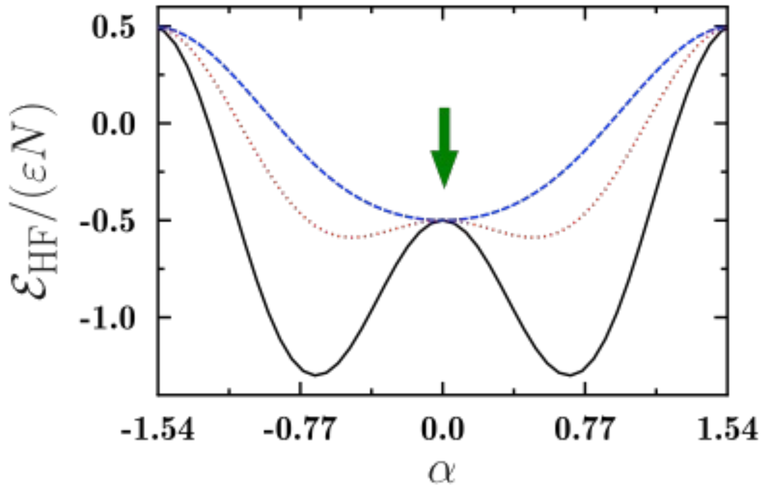


Robin et al, arxiv:2306.16535

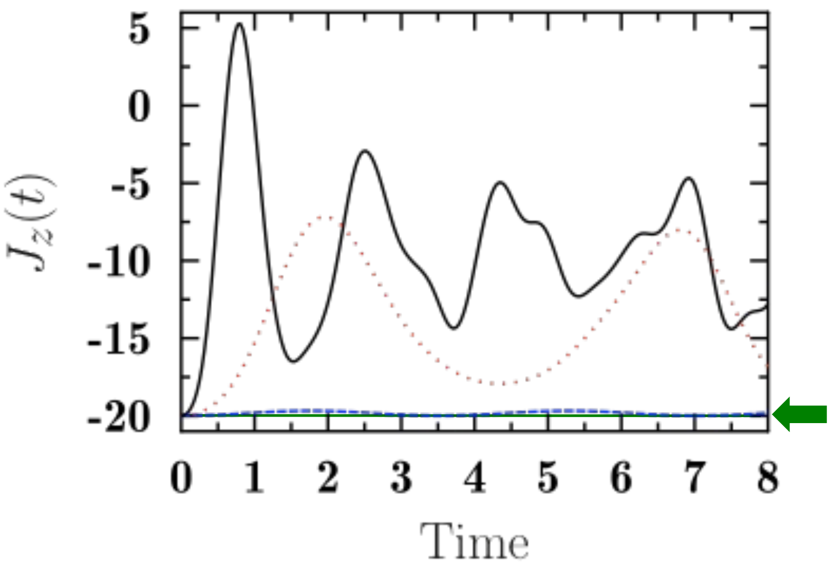
But

$$|\Omega\rangle = \sum_{M=-J}^{J} \sqrt{\binom{2j}{J-M}} \left(\cos\frac{\theta}{2}\right)^{J+M} \left(\sin\frac{\theta}{2}\right)^{J-M} e^{i(J-M)\phi}\,|J, M\rangle$$

N=40 particles



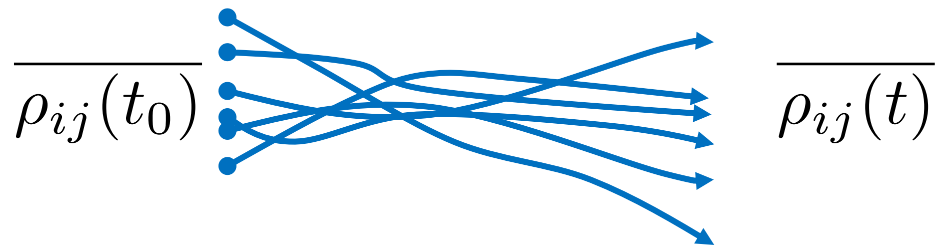
Exact dynamics



Mean-field is stationary

Incorporating correlations through noise

Phase-Space methods

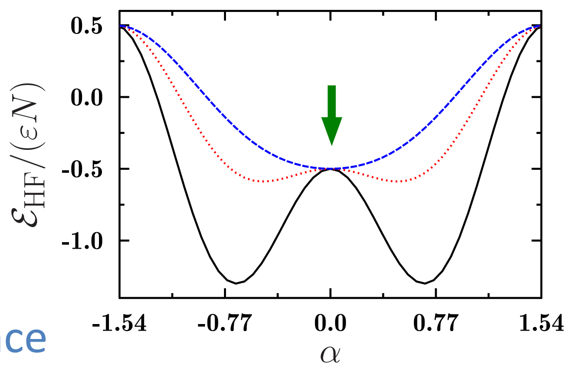


Quantum Jumps

Quantum Monte-Carlo

Benchmarking phase-space method for fermions

$\rho \Rightarrow \rho^{\lambda}$



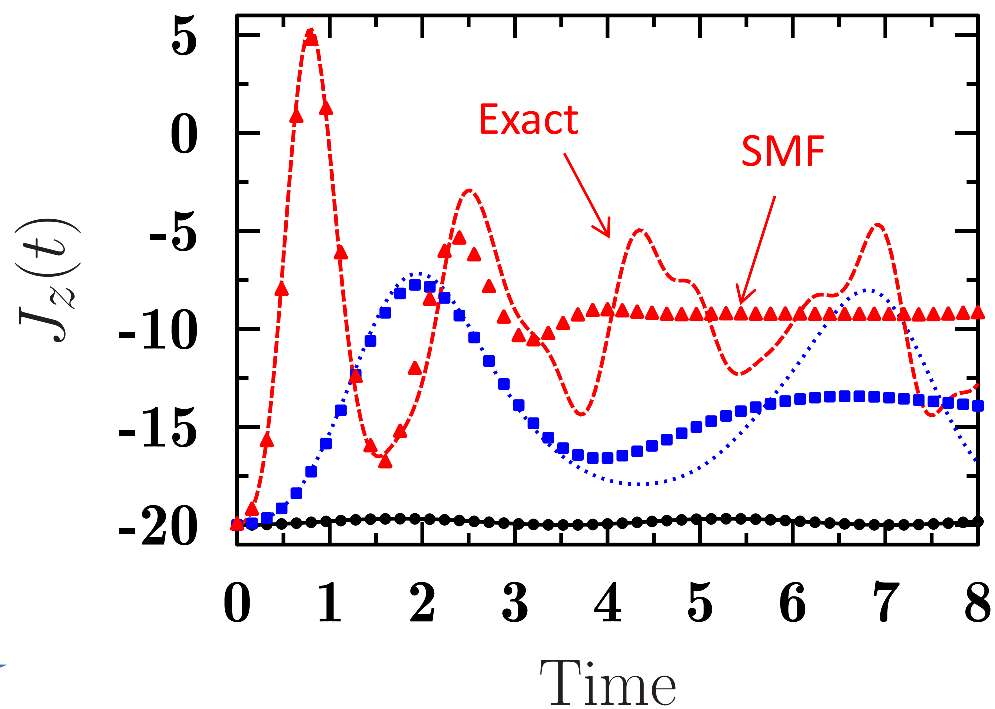
One-body observables

Formulation in quasi-spin space

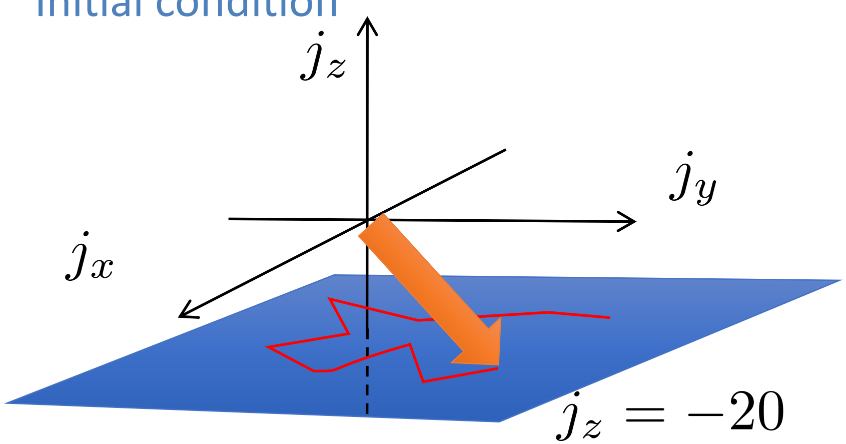
$$j_i \equiv \langle J_i \rangle / N \quad \Rightarrow \quad j_i^{\lambda}$$

$$\overline{j_i^{\lambda}(t_0)} = 0$$

$$\overline{j_x^{\lambda}(t_0) j_x^{\lambda}(t_0)} = \overline{j_y^{\lambda}(t_0) j_y^{\lambda}(t_0)} = \frac{1}{4N}.$$



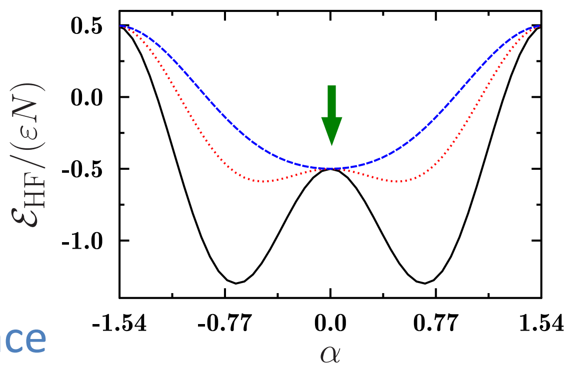
Initial condition



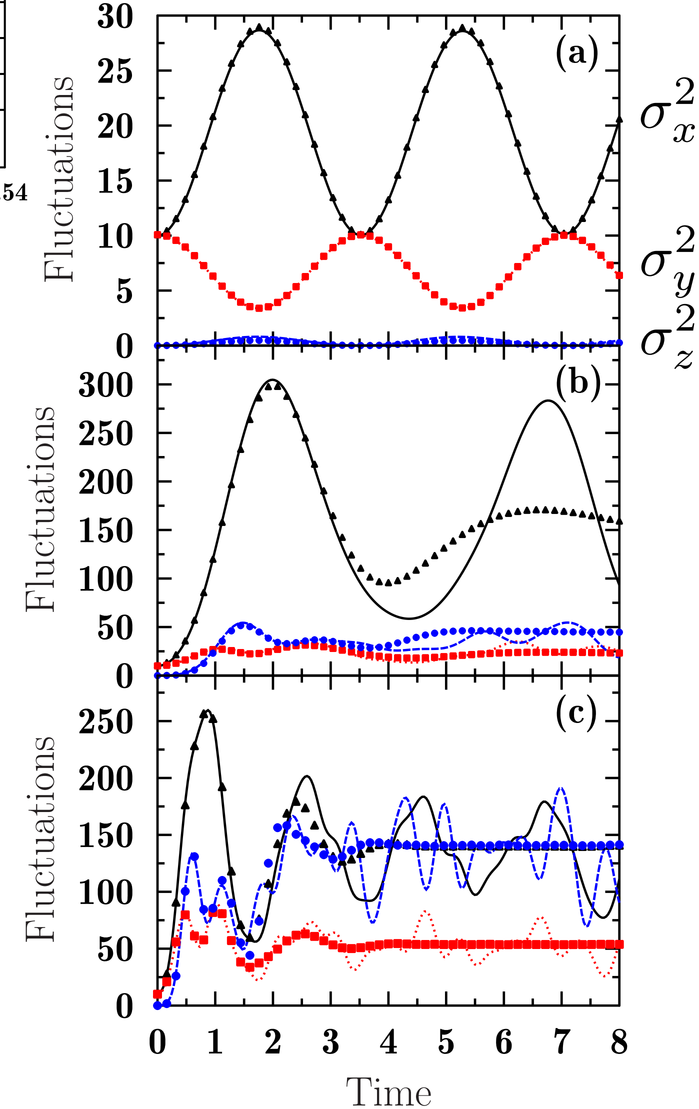
Lacroix, Ayik, Yilmaz, PRC 85 (2012)

Benchmarking phase-space method for fermions

ρ ➡ ρ^{λ}



Fluctuations



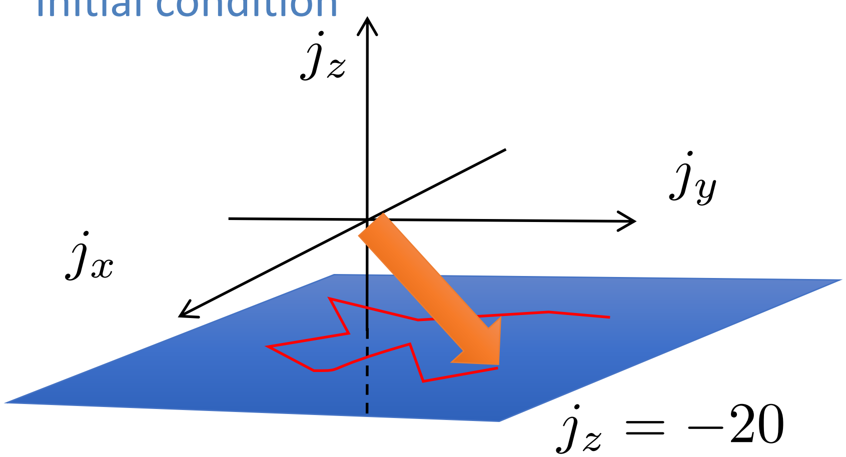
Formulation in quasi-spin space

$j_i \equiv \langle J_i \rangle / N$ ➡ j_i^{λ}

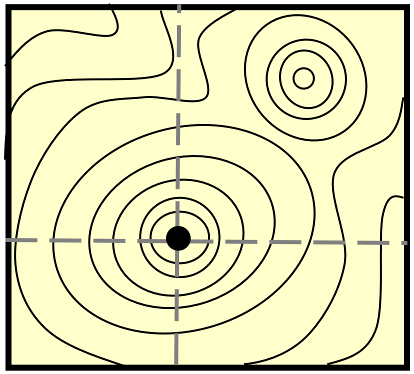
$$\overline{j_i^{\lambda}(t_0)} = 0$$

$$\overline{j_x^{\lambda}(t_0) j_x^{\lambda}(t_0)} = \overline{j_y^{\lambda}(t_0) j_y^{\lambda}(t_0)} = \frac{1}{4N}.$$

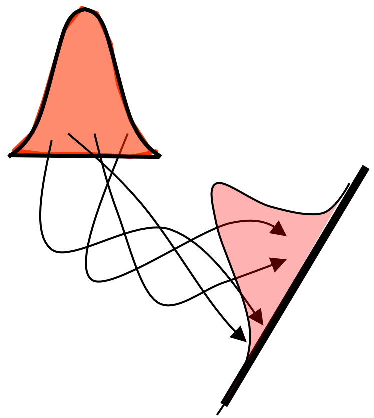
Initial condition



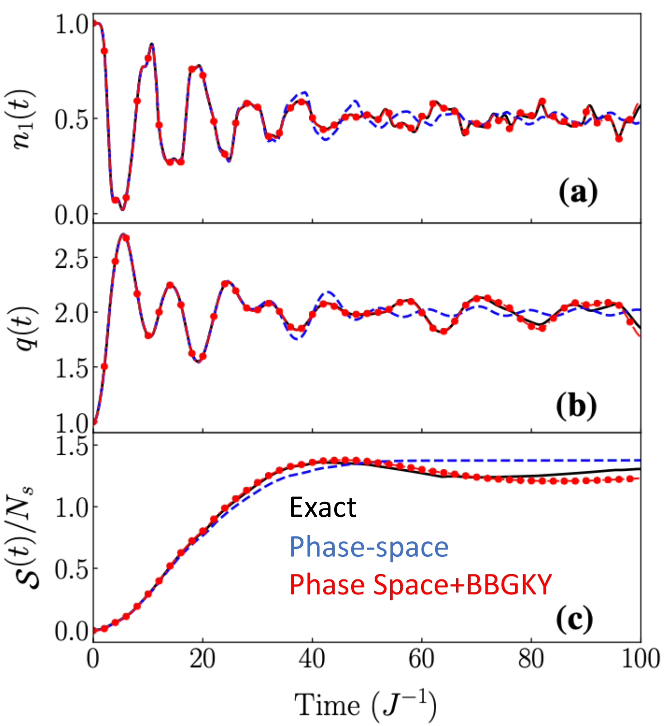
Collective phase-space



Quantum fluctuations



Hubbard model



Applications to toy Hamiltonians

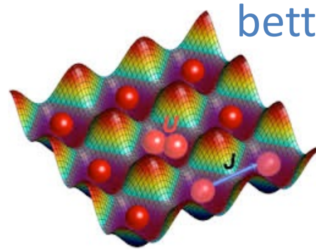
- Extension to superfluid systems: pairing model TDHFB with fluctuations

Lacroix, Gambacurta, Ayik, Yilmaz, PRC C 87, 061302(R) (2013)

- Mapping initial fluctuations with complex Initial correlations

Yilmaz, Lacroix, Curecal, PRC C 90, 054617 (2014).

- Application to Hubbard (1D,2D,3D) model: better than non-equilibrium 2-body green func.



Lacroix, Hermanns, Hinz, Bonitz, PRB90 (2014)

- Equivalent to simplified un-truncated BBGKY hierarchy

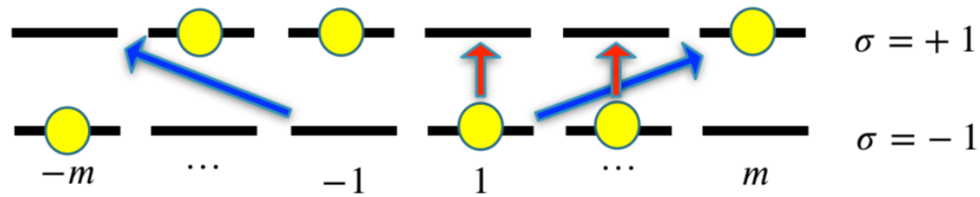
Lacroix, Tanimura, Ayik, EPJA52 (2016)

- Combining BBGKY and phase-space (Hubbard)

Czuba, Lacroix, Regnier, Ulgen, Yilmaz, EPJA56 (2020)

(slide adapted from L. Robledo-ESNT workshop 2023)

THE AGASSI MODEL



- It simulates the behaviour between degenerated energy levels between the Fermi surface
- Parity symmetry
- Number of particles symmetry

$$H = \epsilon J_0 - \frac{1}{2}V(J_+^2 + J_-^2) - g\sum_{\sigma,\sigma'} A_\sigma^\dagger A_{\sigma'}$$

Monopole-monopole interaction: for a given value, there is a QPT that breaks parity in the upper level

Pairing interaction: for a given value, there is a QPT that breaks particle number

The pairing interaction adds a superconducting phase

V and g act as order parameters

$$J_0 = \frac{1}{2}\sum_{\sigma,m} \sigma c_{\sigma,m}^\dagger c_{\sigma,m} \qquad J_+ = J_-^\dagger = \sum_m c_{1,m}^\dagger c_{-1,m} \qquad A_\sigma = \sum_{m>0} c_{\sigma,-m} c_{\sigma,m}$$

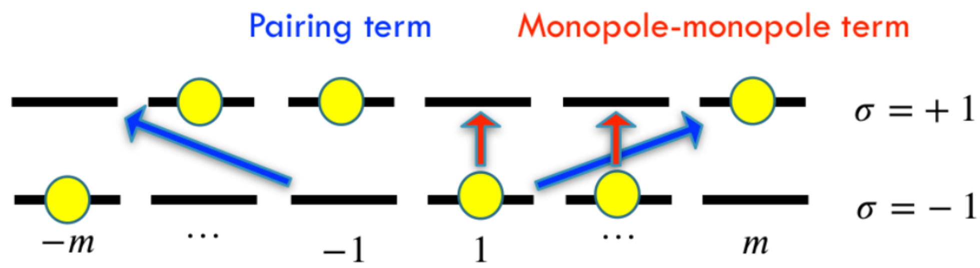
O(5) generators

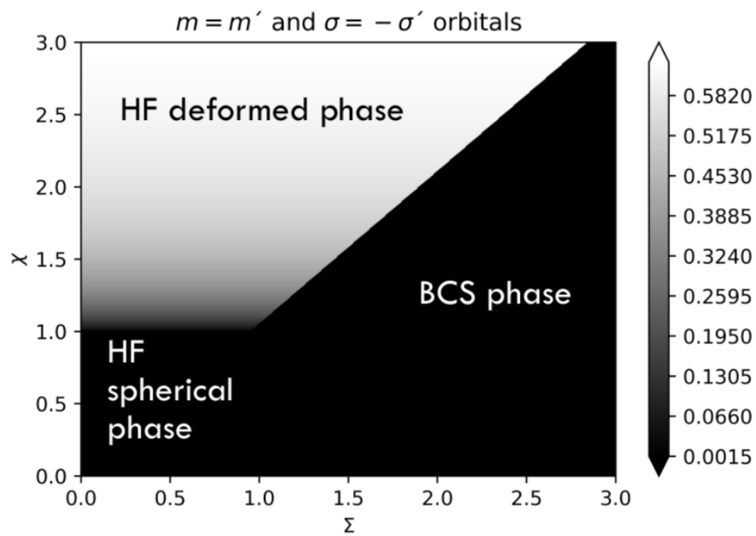
SU(2) generators without pairing interaction (2-Lipkin model)

The HFB ground state has three quantum phases, corresponding to each term

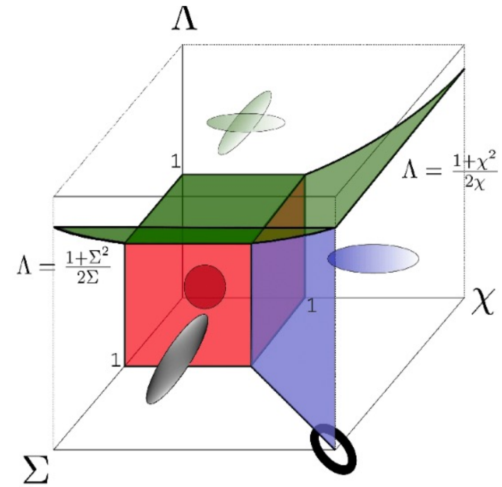
Extensions of the Lipkin model: the Agassi model = Lipkin + Pairing

THE AGASSI MODEL





A test case for studying quantum-Phase-transition



Order parameters

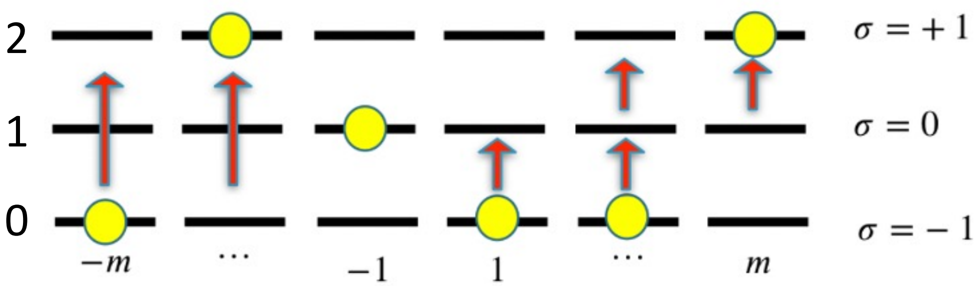
$$\chi = \frac{V}{\epsilon(N-1)} \qquad \Sigma = \frac{g}{\epsilon(N-1)}$$

➡ Link between QPT and entanglement

Fabian, Martin, Robledo, PRA 104 (2021)

➡ Recently used to test Machine Learning and Quantum Machine Learning for QPT

Saiz, et al, PRC 106 (2022)



It essentially adds a level while preserving
The permutation invariance.

$$n_0 + n_1 + n_2 = N$$

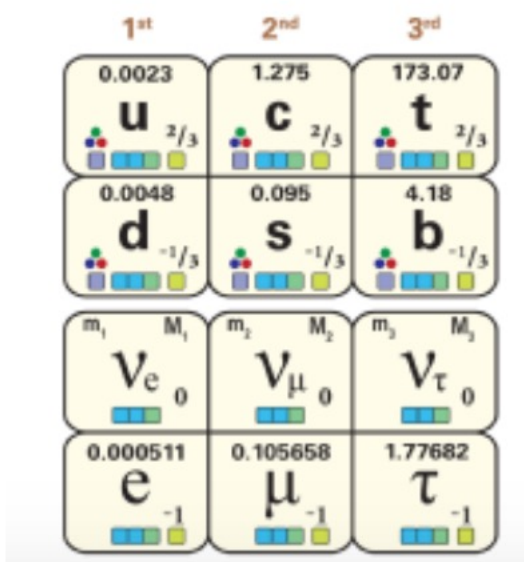
State can be labelled as $|n_1, n_2\rangle$

Hilbert space size: $(N+1)(N+2)/2$

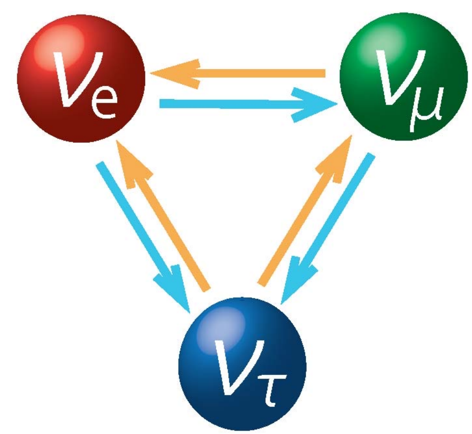
→ SU(3) symmetry

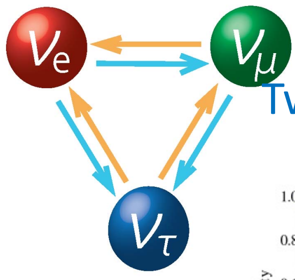
Some interesting physics case

Quark symmetries

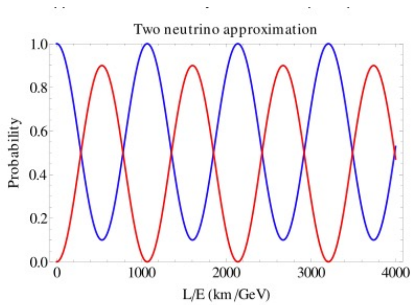


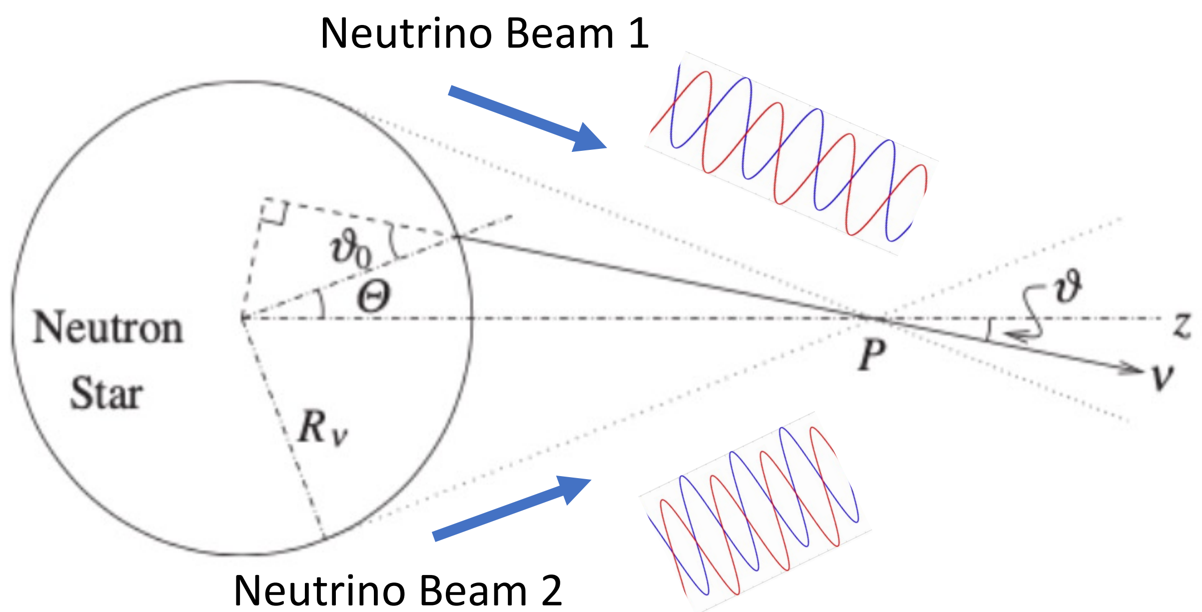
Neutrino flavors oscillations





Two Neutrino oscillat





Hamiltonian

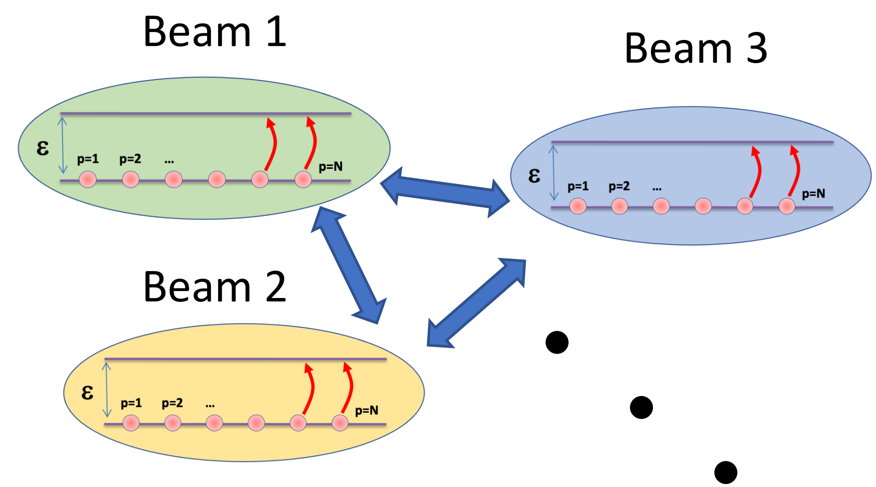
PHYSICAL REVIEW D **84**, 065008 (2011)

Invariants of collective neutrino oscillations

Y. Pehlivan,[1,2,*] A. B. Balantekin,[3,†] Toshitaka Kajino,[2,4,‡] and Takashi Yoshida[4,§]

$$H = \sum_{\omega} \omega \vec{B} \cdot \vec{J}_{\omega} + \mu \sum_{\mathbf{p},\mathbf{q}} (1 - \cos\vartheta_{\mathbf{pq}}) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}.$$

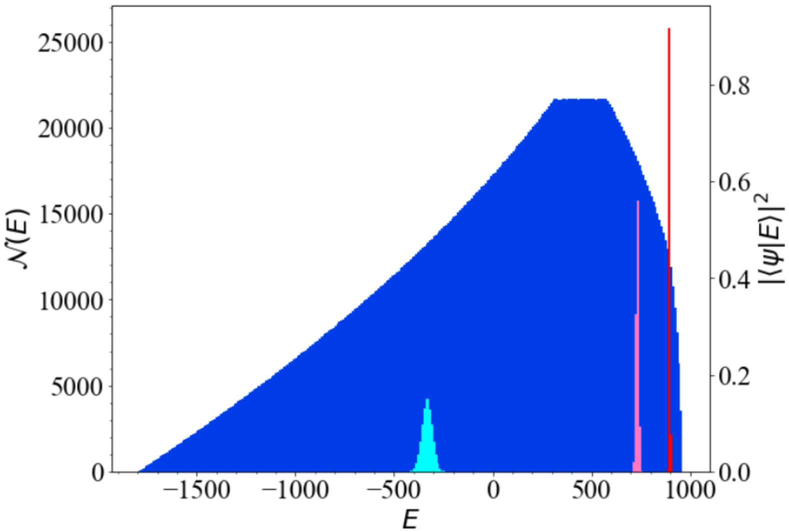
One-body Two-body



Like a set of interacting neutrino beams

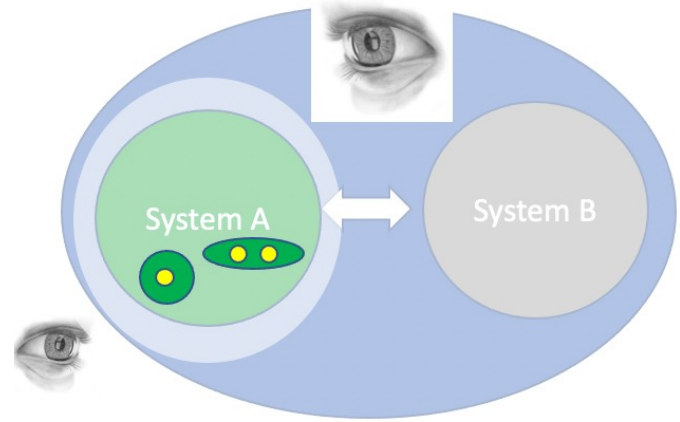
Exact versus Phase-space solution

Exact solution still doable for 2 beams but much more demanding

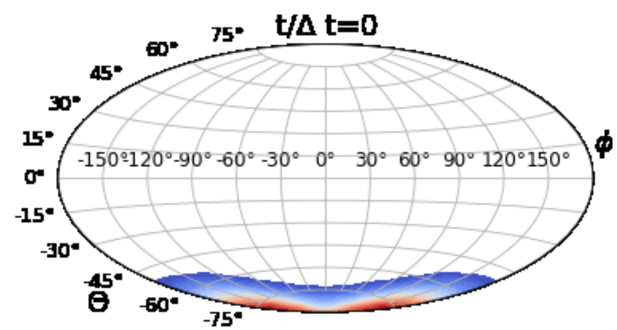


Martin et al, PRD 105 (2022)

Phase-space and entanglement

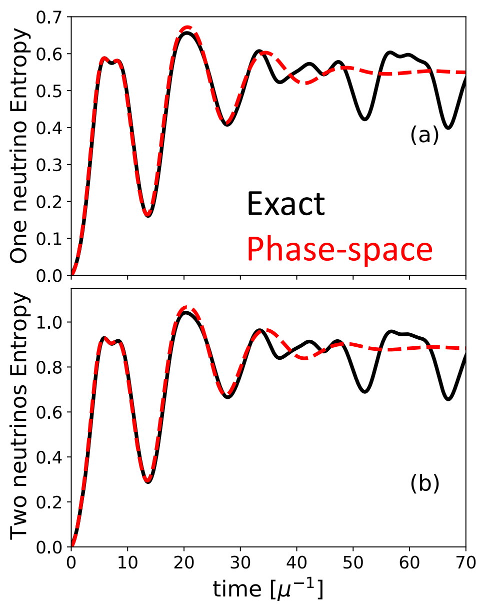


System A – Husimi distribution



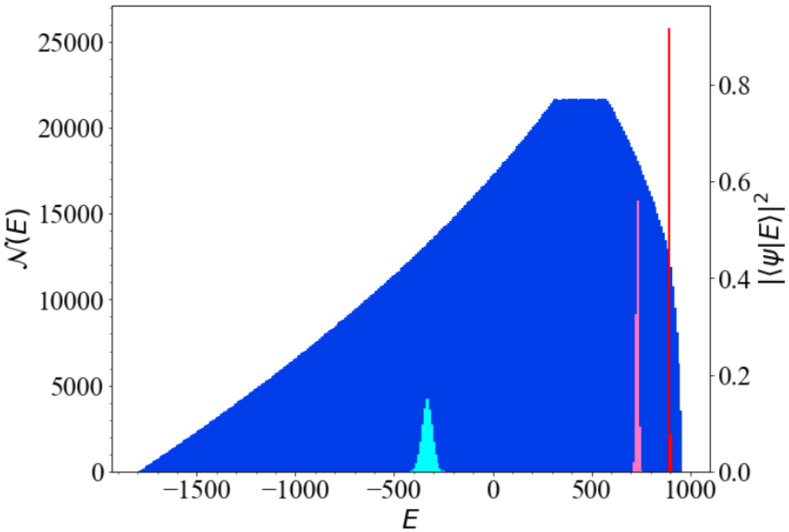
Lacroix et al, PRD 106 (2022)

More Beam ?

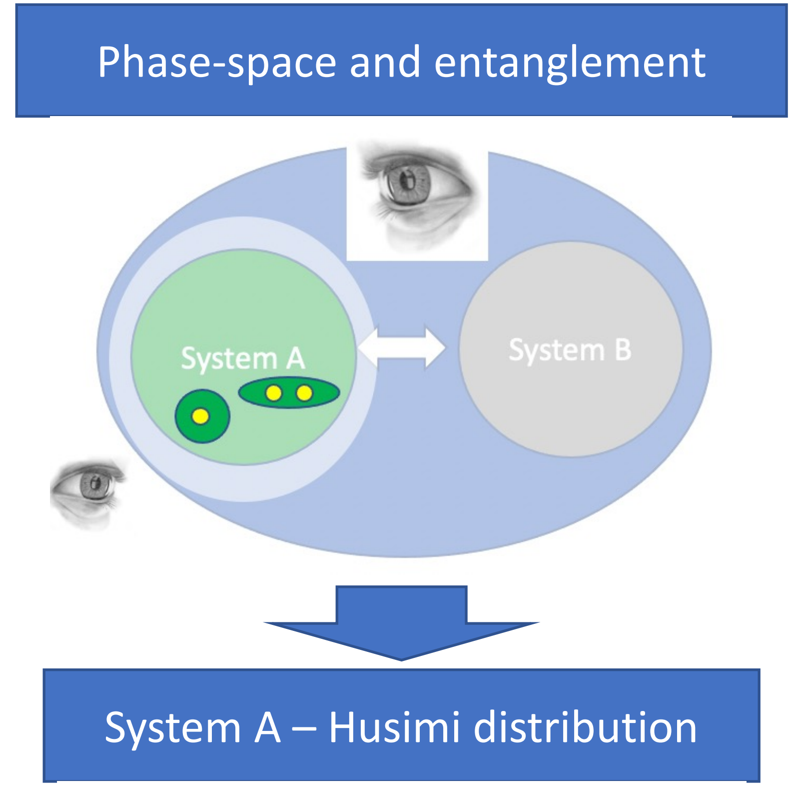


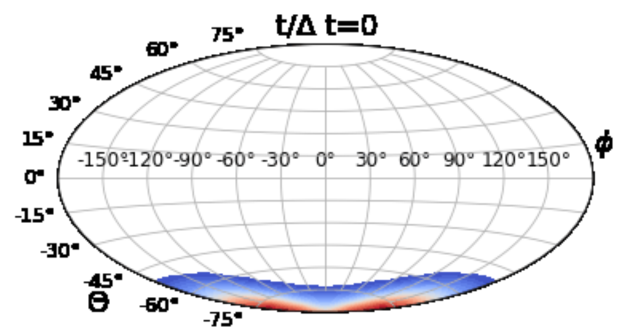
Exact versus Phase-space solution

Exact solution still doable for 2 beams but much more demanding

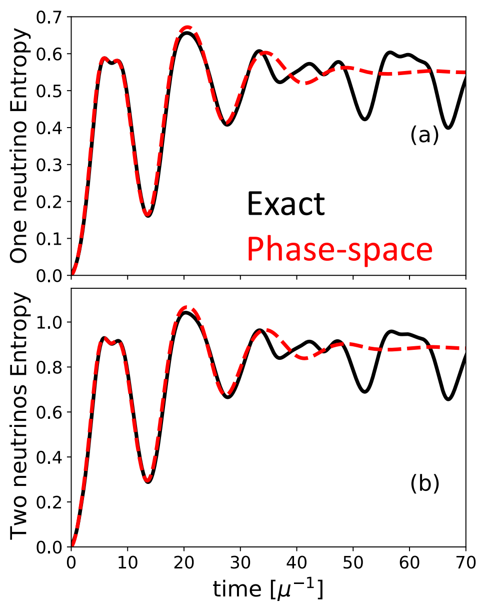


Martin et al, PRD 105 (2022)



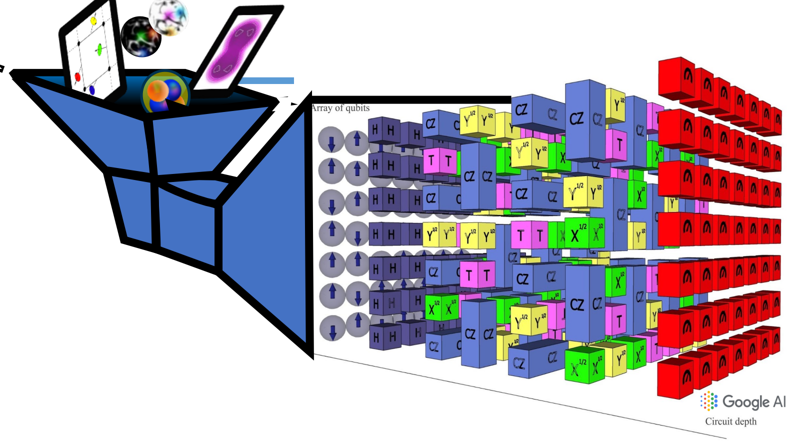


Lacroix et al, PRD 106 (2022)



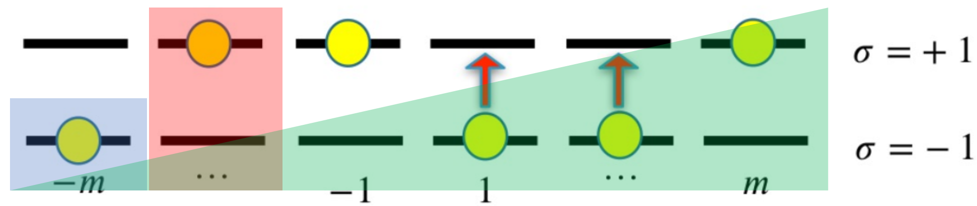
More Beam ?

Quantum computing the Lipkin model



q = Number of qubits

Encoding the Lipkin model on a quantum register

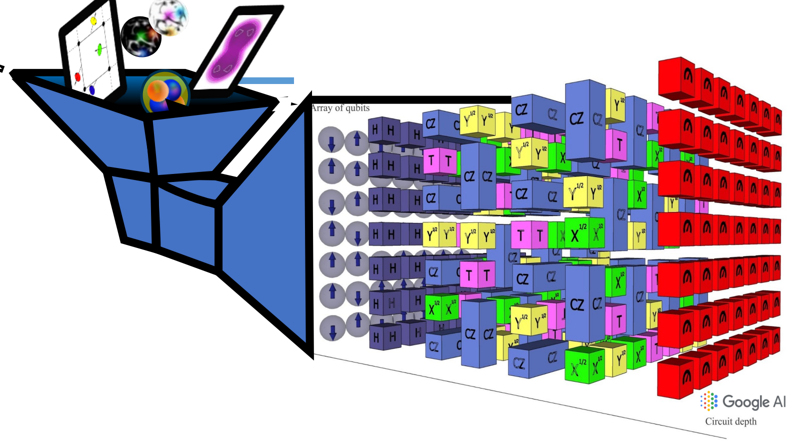


Fermions-to-qubit: Jordan-Wigner	SU(2) encoding	J-scheme (compact) +parity encoding
1 level = 1 qubit	2 levels = 1 qubit	$\vert J, M\rangle \rightarrow \vert[M]\rangle$
$q = 2N$	$q = N$	Use first quantization
		$q = \lceil \log_2 N \rceil$

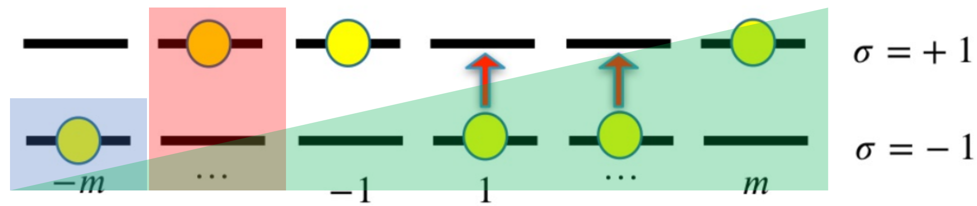
$$\begin{cases} X_\alpha = |1_\alpha\rangle\langle 0_\alpha| + |0_\alpha\rangle\langle 1_\alpha|, \\ Y_\alpha = i|1_\alpha\rangle\langle 0_\alpha| - i|0_\alpha\rangle\langle 1_\alpha|, \\ Z_\alpha = |0_\alpha\rangle\langle 0_\alpha| - |1_\alpha\rangle\langle 1_\alpha| \end{cases}$$

$$H = \frac{\varepsilon}{2}\sum_{\alpha=1}^{N} Z_\alpha + \frac{V}{4}\sum_{\alpha\neq\beta}^{N}(X_\alpha X_\beta - Y_\alpha Y_\beta),$$

Quantum computing the Lipkin model



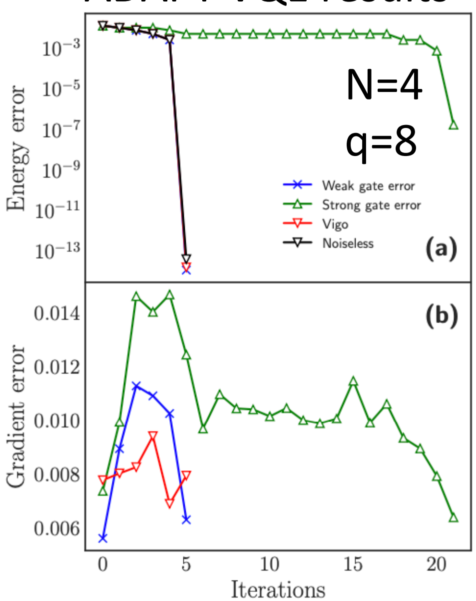
Encoding the Lipkin model on a quantum register



q = Number of qubits

Fermions-to-qubit: Jordan Wigner

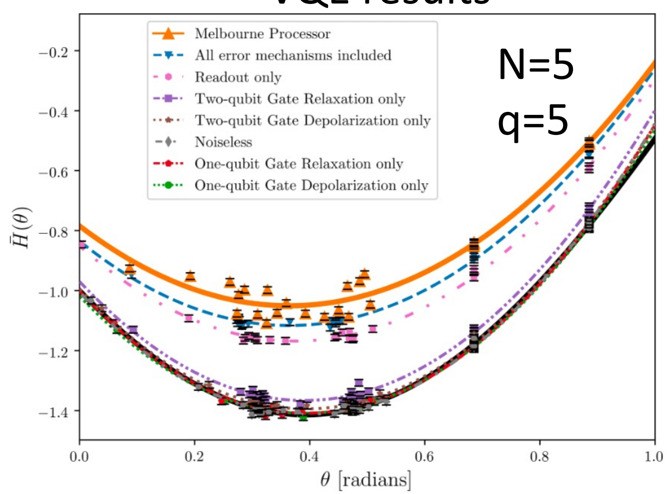
ADAPT-VQE results



J. Romero et al, PRC 105 (2022)

SU(2) encoding

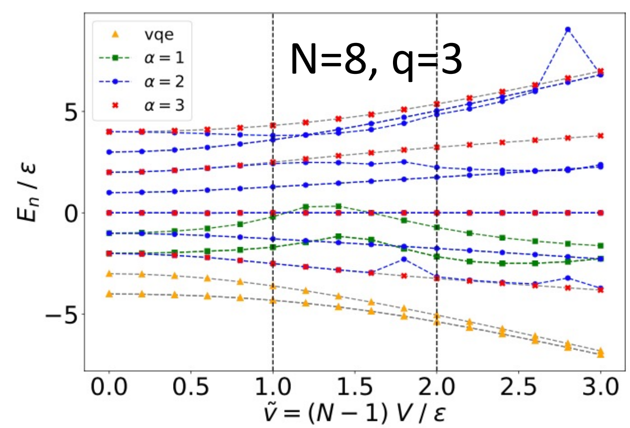
VQE results



M. Cervia et al, PRC 104 (2021)

J-scheme (compact) +parity encoding

QEOM-technique

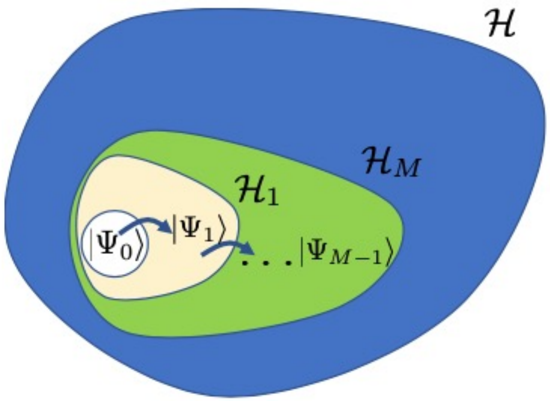


Hlatshwayo et al, PRC 106 (2022), & arXiv:2309.10179

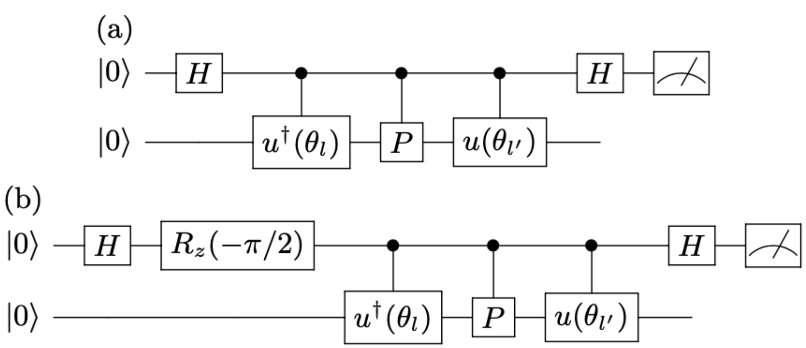
Solving the Lipkin model with using 2 qubits only with hybrid quantum-classical method

Classical post processing

Quantum Subspace expansion



Real/Imaginary parts requires 2 qubits



Quantum Generator Coordinate Method

$$|\Psi\rangle = \int_{\mathbf{q}} f(\mathbf{q})|\Phi(\mathbf{q})\rangle d\mathbf{q}$$

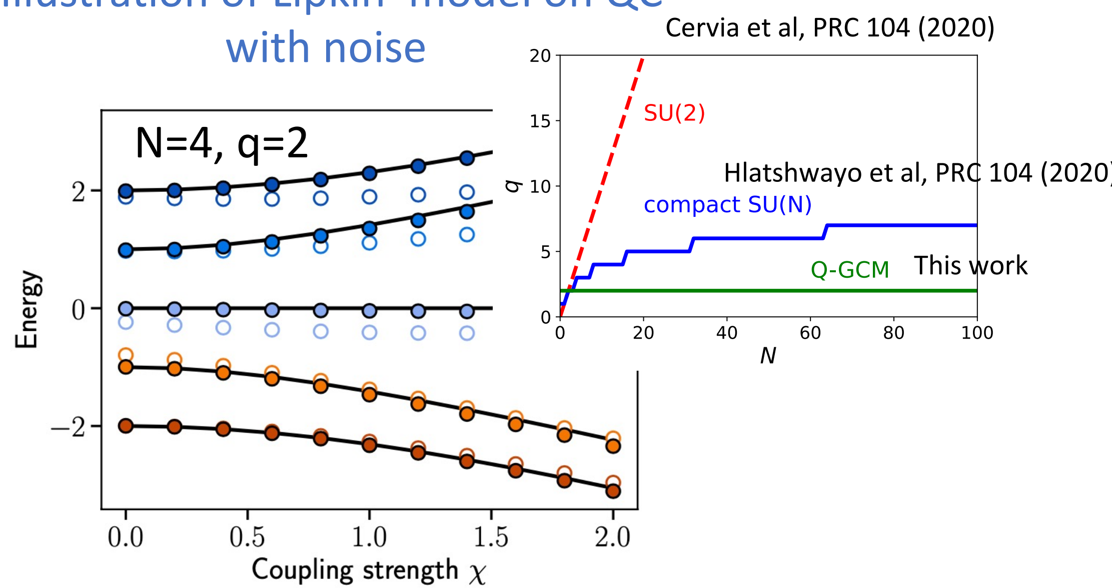
$$\int_{\mathbf{q}'} d\mathbf{q}' \left[\mathcal{H}(\mathbf{q},\mathbf{q}') - E\mathcal{N}(\mathbf{q},\mathbf{q}')\right] f(\mathbf{q}') = 0$$

Lipkin /perm. invariant

$$\langle H\rangle_{ll'} = \frac{\varepsilon N}{2} i_{ll'}^{N-2}\left[i_{ll'} z_{ll'} + \frac{\chi}{2}\left(x_{ll'}^2 - y_{ll'}^2\right)\right]$$

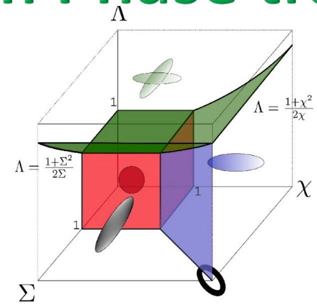
one-body kernels

Illustration of Lipkin model on QC with noise

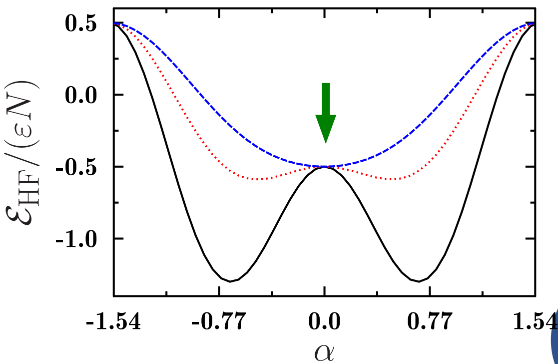


Y. Beaujeault-Taudière and DL, arxiv (2023)

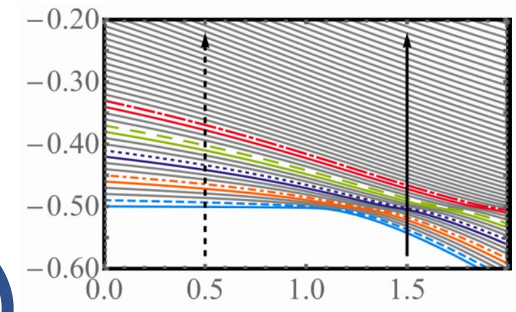
Quantum Phase transitions

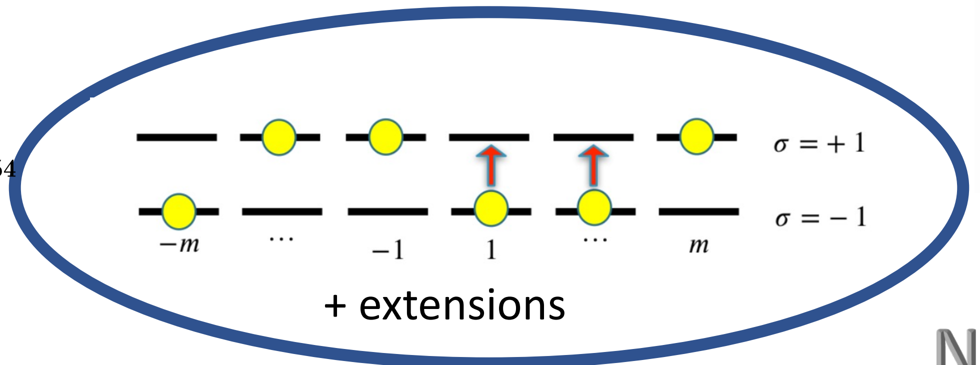


Role of Symmetries

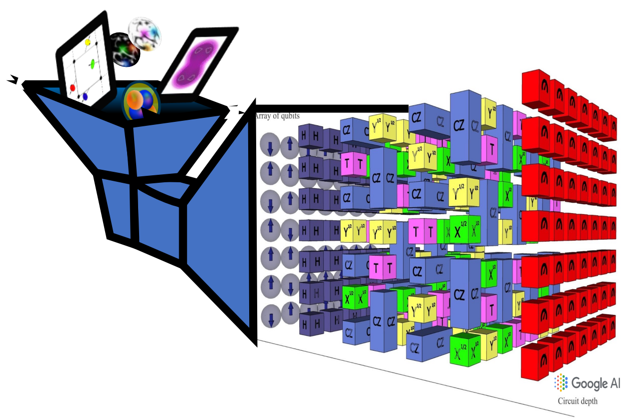


Eigenvalue Properties

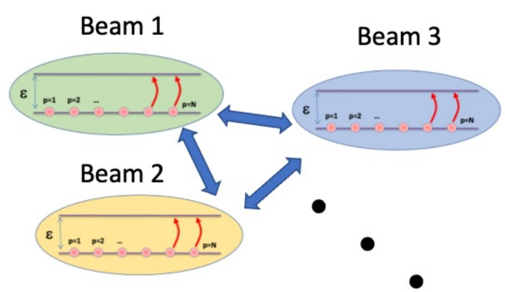




Testing quantum computers



Towards neutrino physics



Non-equilibrium